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ON ELASTIC PLASTIC DEFORMATION

IN BEAMS UNDER DYNAMIC LOADING

by

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On Elastic Plastic Deformation
in Beams under Dynamic Loading[1]
by J. A. Seiler[2]

Abstract: This paper gives an elastic-plastic analysis of a
simply supported uniform beam subjected to a uniform pressure
applied as a pulse of rectangular shape. Plastic flow is taken
account of only at a plastic hinge at the mid-section of the
beam. The resulting permanent deformations are compared with
those predicted by a "rigid-plastic" type of analysis in which
elastic deformations are neglected. Because of the failure of
the elastic-plastic analysis to consider the plastic deformations
at cross-sections other than the middle section, the two solu-
tions do not agree even at large load values. The elastic-
plastic results are in better agreement with an incorrect rigid-
plastic treatment in which plastic hinge action is assumed to
occur only at the mid-point.

I. Introduction

There has recently been considerable study of the plastic
deformations of beams and frames under dynamic loading, on the
basis of the "plastic-rigid" hypothesis [1, 2, 3, 4, 5, 6][3].

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3. Numbers in square brackets refer to the bibliography at the
    end of the paper.

According to this hypothesis there is no curvature change at a
given section of a beam unless a bending moment of a certain
magnitude is maintained at that section. Where this moment
(called the limit or fully plastic moment) is maintained, "plas-
tic hinge" action can occur as long as the bending moment main-
tains its limit value. The neglect of elastic deformations is
permissible when the loads are of such a magnitude as to produce
plastic strains large in comparison with any possible elastic
strains.

In order to determine the range of loading in which the
neglect of elastic deformations is permissible, it is desirable
to make analytical studies in which elastic as well as plastic
strains are considered. One study of this type has been carried
out by Bleich and Salvadori [7] for a problem of impulsive
motion (specified initial velocity) of a uniform beam with free
ends. The present paper treats the elastic and plastic motions
of a simply supported beam subjected to a specified dynamic
load, namely a uniformly distributed pressure applied as a
pulse of rectangular shape. The analysis used here is essen-
tially the same as that of [7]. The initial elastic motion is
represented as a series of eigenfunctions. The initial wholly
elastic phase terminates when the bending moment at any section
reaches the limit moment magnitude. Thereafter at such a sec-
tion the moment is held constant at the limit moment magnitude
and the previous condition of slope continuity is relaxed. Thus
"plastic hinge" action occurs at such a section as long as the
relative angular velocities across the section are in the same

sense as the moment at the hinge. The subsequent elastic motions of the segments separated by the hinge section are then represented by a new series of eigenfunctions corresponding to the new boundary conditions, with amplitudes chosen so that the elastic-plastic solution matches the wholly elastic one.

The plastic-rigid solutions of beam problems show that there are in general not only plastic hinges at fixed cross-sections, but also either travelling plastic hinges or finite plastic regions of changing size. Although the solution of elastic-plastic problems by synthesizing eigenfunctions can clearly be carried out when one or more fixed plastic hinges are present, this method is not applicable when moving hinges or elastic-plastic interfaces occur. Thus in [7] it is assumed in the numerical example treated that plastic hinge action occurs only at the center of the beam. This is certainly correct for low enough initial velocities and incorrect for very high initial velocities. Unfortunately, since the plastic-rigid type of analysis requires the initial velocities to be high in order for the analysis to be appropriate, it is evident that the validity of the plastic-rigid analysis cannot be assessed by comparison with the results of a "single-hinge" type of elastic-plastic analysis. In [7] Bleich and Salvadori compared the deformations of their elastic-plastic solution with those of a fictitious rigid-plastic solution in which a plastic hinge was assumed to occur only at the center, the two halves of the beam moving as rigid bars. Their elastic-plastic results were found to coincide at high initial velocities with those of this

"single-hinge" rigid-plastic solution. This is clearly to be expected since both solutions would be correct for a bar sufficiently strengthened at sections other than the middle section.

The simplest case to which the above method applies is the one degree of freedom system, illustrated by a mass on a spring. This problem has been thoroughly worked out by Brooks and Newmark [8], and will be described here to illustrate both the general method of elastic-plastic analysis and the expected relation between the results of this analysis and those of the simpler rigid-plastic analysis.

Consider the system shown in Fig. 1. The spring force-displacement curve, according to the above method, is taken as shown in Fig. 2, and for simplicity, we consider a rectangular pulse shape as shown in Fig. 3.

If x is the displacement of the mass, p is the load intensity, and k is the spring constant, then the equation of motion of the mass m, as the load is first applied is

$$m\ddot{x} = p - kx \tag{1}$$

subject to

$$x(0) = \dot{x}(0) = 0, \tag{2}$$

where the dot denotes time differentiation. The system (1), (2) has the solution

$$x = \frac{p}{k}\left(1 - \cos\sqrt{\frac{k}{m}}\,t\right). \tag{3}$$

The system reaches its maximum elastic response when $x = x_y$, the yield displacement, which occurs at time $t = t_y$.

given by

$$\sqrt{\frac{k}{m}} \; t_y = \text{arc-cos} \left(\frac{\mu - 1}{\mu} \right), \qquad (4)$$

where we define the dimensionless parameter μ to be

$$\mu \equiv \frac{\dot{p}}{kx_y} \; .$$

At time $t = t_y$ we have

$$x = x_y$$

$$\dot{x} = \frac{p}{k} \; \sqrt{\frac{k}{m}} \; \frac{\sqrt{2\mu - 1}}{\mu}, \qquad (5)$$

which are used as initial conditions for the new equation of
motion of the mass in the plastic range, i.e.,

$$m\ddot{x} = p - kx_y. \qquad (6)$$

Equation (6), together with initial conditions (5) has solution

$$x = \frac{p - kx_y}{2m} \; (t - t_y)^2 + \frac{p}{k} \; \sqrt{\frac{k}{m}} \; \frac{\sqrt{2\mu - 1}}{\mu} \; (t - t_y) + x_y. \qquad (7)$$

At time τ the load ends, at which time the velocity and dis-
placement of the mass are:

$$x = \frac{p - kx_y}{2m} \; (\tau - t_y)^2 + \frac{p}{k} \; \sqrt{\frac{k}{m}} \; \frac{\sqrt{2\mu - 1}}{\mu} \; (\tau - t_y) + x_y \qquad (8)$$

$$\dot{x} = \frac{p - kx_y}{m} \; (\tau - t_y) + \frac{p}{k} \; \sqrt{\frac{k}{m}} \; \frac{\sqrt{2\mu - 1}}{\mu} \; .$$

The equation of motion of the mass now becomes

$$m\ddot{x} = - kx_y, \qquad (9)$$

subject to (8) as initial conditions. The system (8), (9) has
solution

$$x = - \frac{kx_y}{2m} (t - \tau)^2 + \left[\frac{p - kx_y}{m} (\tau - t_y) + \frac{p}{k} \sqrt{\frac{k}{m}} \ \frac{\sqrt{2\mu - 1}}{\mu} \right] (t - \tau)$$

$$+ \frac{p - kx_y}{2m} (\tau - t_y)^2 + \frac{p}{k} \frac{\sqrt{2\mu - 1}}{\mu} (\tau - t_y) + x_y.$$

$$(10)$$

To find the maximum response of the system, we set $\dot{x} = 0$ and solve for t, insert this back in (10) and obtain an expression for the final maximum response x_F:

$$x_F = \frac{kx_y \tau^2}{m} \left[\frac{\mu(\mu - 1)}{2} \left(\frac{\tau - t_y}{\tau} \right)^2 + \mu \sqrt{2\mu - 1} \ \sqrt{\frac{m}{k}} \left(\frac{\tau - t_y}{\tau^2} \right) \right.$$

$$\left. + \frac{m}{k\tau^2} \left(\frac{2\mu + 1}{2} \right) \right].$$

$$(11)$$

We now define a dimensionless pulse time $\zeta = \sqrt{k/m} \ \tau$, in terms of which (11) gives the dimensionless maximum displacement X_F as

$$(X_F)_{E.P.} \equiv \frac{mx_F}{kx_y \tau^2} = \frac{\mu(\mu - 1)}{2} \left[1 - \frac{1}{\zeta} \text{arc-cos} \left(\frac{\mu - 1}{\mu} \right) \right]^2$$

$$+ \frac{\mu \sqrt{2\mu - 1}}{\zeta} \left[1 - \frac{1}{\zeta} \text{arc-cos} \left(\frac{\mu - 1}{\mu} \right) \right] + \frac{2\mu + 1}{2\zeta^2}, \qquad (12)$$

where the arc-cos function is restricted, of course, to its principle value.

It is evident that for low enough load intensity and/or short enough pulse time, we might not have yielding, or that yielding might occur after the load is removed. The analysis in this case is much the same and will not be presented here. The resulting dimensionless deformation is

$$(X_F)_{E.P.} \equiv \left(\frac{mx_F}{kx_y \tau^2} \right) = \frac{1}{\zeta^2} \left[\mu^2 (1 - \cos \zeta) + \frac{1}{2} \right]. \qquad (13)$$

ζ here is restricted to have a small enough value so that yield-
ing will not occur during the force pulse.

Consider now the plastic-rigid problem, i.e., with a
spring-force displacement curve as shown in Fig. 4. In this
case a similar analysis yields the following expression for the
dimensionless maximum displacement:

$$(X_F)_{R.P.} = \frac{1}{2}\mu(\mu - 1).\tag{14}$$

Figure 5 shows curves of $(X_F)_{E.P.}$ vs load intensity μ
for various ζ, and also the curve of $(X_F)_{R.P.}$ vs. μ.

The criterion for the validity of a plastic-rigid analy-
sis of this problem, following that put forth in [1], is that
we must have

$$kx_y(X_F)_{R.P.} \gg \frac{1}{2}kx_y^2.\tag{15}$$

Relation (15) states that the plastic work (i.e., yield force
multiplied by the plastic deformation) must be much greater than
the maximum elastic strain energy that can be stored in the
spring. Relation (15) can also be written in the following form:

$$\frac{W_P}{W_E} = \zeta^2\mu(\mu - 1) \gg 1,\tag{16}$$

where W_P and W_E are, respectively, the plastic work and the
maximum elastic strain energy.

It is seen from Figs. 6(a), (b), (c) that the validity
of the rigid-plastic analysis of this problem cannot be ascer-
tained from the magnitude of the energy ratio W_P/W_E alone. The
validity must also depend upon an inequality being satisfied by

one of the parameters μ, ζ. In other terms, one cannot make a
positive statement that W_P/W_E being greater than some number
implies that the error involved in the rigid-plastic hypothesis
is less than some preassigned value. The statement that can be
made is that given a fixed ratio of pulse time to period, a
value of W_P/W_E can be found so that the error is less than any
preassigned amount.

Figure (6) shows plots of percent error vs. μ for various
W_P/W_E values. Figure 6 may be interpreted as follows. If, say,
$W_P/W_E = 10$, and $\mu = 13$, then the error involved in the plastic-
rigid assumption will always be less than 15% provided the pulse
time is larger than the value obtained from (16) namely
$\zeta = .253$, which implies a pulse time of .040 periods.

II. Elastic Response of Beams

We now proceed to the discussion of a simply supported
beam of mass m per unit length, and length ℓ as indicated in
Fig. 7, acted on by a uniform load of intensity p per unit
length. We again consider only the rectangular force pulse
shape as in Fig. 3. The equation for the displacement y (x,t)
of the beam is, (neglecting the effects of shear and rotary
inertia)

$$(EIy_{xx})_{xx} + my_{tt} = p \, . \tag{17}$$

The initial and boundary conditions are

$$\begin{rcases} y(x,0) = y_t(x,0) = 0 \\ y(0,t) = y_{xx}(0,t) = 0 \\ y(\ell,t) = y_{xx}(\ell,t) = 0. \end{rcases} \tag{18}$$

The eigenfunctions of the homogeneous system are

$$\varphi_n(x) = \sin \frac{n\pi x}{\ell} \, ,$$

and we assume a solution in the form

$$y(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)q_n(t), \qquad (19)$$

where each generalized coordinate $q_n(t)$ must satisfy the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) + \frac{\partial V}{\partial q_n} = Q_n, \qquad (20)$$

and where

$$T = \frac{m}{2} \int_0^{\ell} (y_t)^2 dx, \qquad (21)$$

$$V = \frac{EI}{2} \int_0^{\ell} (y_{xx})^2 dx, \qquad (22)$$

$$Q_n = \int_0^{\ell} P(x,t)\varphi_n(x)dx = p \int_0^{\ell} \sin \frac{n\pi x}{\ell} dx. \qquad (23)$$

Upon substitution of (21), (22), (23) into (20) we find

$$q_n(t) = \frac{4p}{nm\pi\lambda^4} (1 - \cos \lambda^2 t), \qquad (24)$$

where

$$n = 1, 3, 5 \ldots$$

$$\lambda^4 \equiv \frac{n^4\pi^4}{k^2\ell^4} , \qquad k^2 \equiv \frac{m}{EI} \, .$$

Hence our solution for $y(x,t)$ becomes

$$y(x,t) = \frac{4\ell^4 p}{\pi^5 EI} \sum_{n=1,3}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{\ell} (1 - \cos \frac{n^2\pi^2}{k\ell^2} t). \qquad (25)$$

This solution is valid until the bending moment reaches its capacity value M_0 at the center of the beam. In order to find the time t_0 at which this occurs we set the bending moment $M(x, t_0) = - EIy_{xx}(x, t_0)$ equal to M_0 at $x = \ell/2$. We obtain

$$M_0 = \frac{4p\ell^2}{\pi^3} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} (1 - \cos \frac{n^2\pi^2 t_0}{k\ell^2}). \qquad (26)$$

We now define dimensionless load and time parameters μ and η to be

$$\mu \equiv \frac{p\ell^2}{M_0}, \qquad \eta = \frac{\pi^2 t}{k\ell^2}.$$

Then (26) becomes

$$\frac{\pi^3}{32} (\frac{\mu - 8}{\mu}) = \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \cos n^2 \eta_0 \qquad (27)$$

where $\eta_0 = \pi^2 t_0 / k\ell^2$.

The right-hand side of Eq. (27) is plotted vs. η_0, and for each μ value the intersection of left- and right-hand sides gives the time for the moment to reach M_0 at $x = \ell/2$. Figure 8 shows a plot of η_0 versus the load parameter μ. At time $\eta = \eta_0$ the velocity and displacement of all points of the beam are given by:

$$y(x, t_0) = \frac{4p\ell^4}{\pi^5 EI} \sum_{n=1,3}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{\ell} (1 - \cos n^2 \eta_0) \qquad (28)$$

$$y_t(x, t_0) = \frac{4p\ell^2}{\pi^3 EIk} \sum_{n=1,3}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{\ell} \sin n^2 \eta_0. \qquad (29)$$

Now a new problem must be solved; we have a beam of length $\ell/2$, simply supported at one end, with zero shear and constant moment M_O at the other end, and with the above displacement and velocity as initial conditions (Fig. 9).

This problem as it stands does not lend itself to solution by a sum of eigenfunctions, since the boundary conditions are non-homogeneous. However, the problem can be separated into two parts. We seek a function $y(x,t')$, where $t' \equiv t - t_o$, such that

$$
\left.
\begin{aligned}
y_{xxxx} + k^2 y_{t't'} &= \frac{p}{EI} \\[2mm]
y(0,t') = y_{xx}(0,t') &= 0 \\[2mm]
y_{xx}(\tfrac{\ell}{2},t') = y_{xxx}(\tfrac{\ell}{2},t') &= 0
\end{aligned}
\right\}
\qquad (30)
$$

to which we add a particular solution satisfying the non-homogeneous boundary conditions of zero shear and constant moment at the free end of the beam. The desired solution is

$$
y(x,t') = (A + Bt')x + \sum_{n=1}^{\infty} \varphi_n(x) \left[A_n \sin \frac{b_n^2 t'}{k\ell^2} + B_n \cos \frac{b_n^2 t'}{k\ell^2} \right.
$$

$$
+ \left. \frac{4p\ell^4}{EIb_n^5 \left(\sinh \frac{b_n}{2} + \sin \frac{b_n}{2}\right)} \right] + \frac{M_o\ell^2}{EI} \left[\frac{\mu x^4}{24\ell^4} \right.
$$

$$
\left. - \frac{x^3(\mu + 24)}{48\ell^3} + \frac{(\mu - 8)}{2} \left(\frac{3x(t')^2}{k^2\ell^5} - \frac{x^5}{20\ell^5} \right) \right]
\qquad (31)
$$

where A, B, A_n, B_n are constants to be determined from the initial conditions, $\varphi_n(x)$ are the eigenfunctions of the half

beam, and b_n are the eigen values:

$$\varphi_n(x) = \sin \frac{b_n}{2} \sinh \frac{b_n x}{\ell} + \sinh \frac{b_n}{2} \sin \frac{b_n x}{\ell}$$

$$\tanh \frac{b_n}{2} = \tan \frac{b_n}{2}, \text{ or}$$

$$b_n \approx \frac{\pi}{2}(4n + 1).$$

Proceeding in the usual way for determining the coefficients, we find:

$$A = \frac{96 p \ell^3}{\pi^7 EI} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^7}(1 - \cos n^2 \eta_0) + \frac{(2\mu + 117)M_0 \ell}{1680 \ EI}$$

$$B = \frac{96 p \ell}{\pi^5 EIk} \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^5} \sin n^2 \eta_0$$

$$A_n = \frac{16 \sqrt{2} \ p \ell^4 (-1)^n \cosh \frac{b_n}{2}}{\pi^3 EI b_n (\sinh^2 \frac{b_n}{2} - \sin^2 \frac{b_n}{2})} \sum_{m=1,3}^{\infty} \frac{(-1)^{\frac{m-1}{2}} \sin m^2 \eta_0}{m(m^4 \pi^4 - b_n^4)}$$

$$B_n = \frac{16 \sqrt{2} \ p \ell^4 b_n (-1)^n \cosh \frac{b_n}{2}}{\pi^3 EI (\sinh^2 \frac{b_n}{2} - \sin^2 \frac{b_n}{2})} \sum_{m=1,3}^{\infty} \frac{(-1)^{\frac{m-1}{2}}(1 - \cos m^2 \eta_0)}{m^3(m^4 \pi^4 - b_n^4)}$$

$$+ \frac{4 M_0 \ell^2}{EI b_n^3 (\sinh^2 \frac{b_n}{2} - \sin^2 \frac{b_n}{2})} \left[\frac{\sin \frac{b_n}{2} \cosh \frac{b_n}{2}(\mu + 8)}{4} \right.$$

$$\left. - \frac{\mu(\sinh \frac{b_n}{2} - \sin \frac{b_n}{2})}{b_n^2} \right] - \frac{4 p \ell^4}{EI b_n^5 (\sinh \frac{b_n}{2} + \sin \frac{b_n}{2})}.$$

This solution applies until

$$t = \tau$$

at which time the load is removed. For convenience we define a dimensionless quantity ζ by $\zeta \equiv \pi^2\tau/k\ell^2$; physically ζ is 2π times the ratio of pulse duration to fundamental period. Then at time $\eta = \zeta$ the velocity and displacement of all points of the beam are given by

$$y(x, \frac{k\ell^2(\zeta - \eta_o)}{\pi^2}) = (A + B \frac{k\ell^2(\zeta - \eta_o)}{\pi^2}) x + \psi(x, \frac{k\ell^2(\zeta - \eta_o)}{\pi^2})$$

$$+ \sum_{n=1}^{\infty} \varphi_n(x)[A_n \sin \frac{b_n^2(\zeta - \eta_o)}{\pi^2} + B_n \cos \frac{b_n^2(\zeta - \eta_o)}{\pi^2}$$

$$+ \frac{4p\ell^4}{EIb_n^5(\sinh \frac{b_n}{2} + \sin \frac{b_n}{2}}]. \quad (32)$$

$$y_{t'}(x, \frac{k\ell^2(\zeta - \eta_o)}{\pi^2}) = Bx + \dot\psi(x, \frac{k\ell^2(\zeta - \eta_o)}{\pi^2}$$

$$+ \sum_{n=1}^{\infty} \frac{\varphi_n(x)b_n^2}{k\ell^2}[A_n \cos \frac{b_n^2(\zeta - \eta_o)}{\pi^2} - B_n \sin \frac{b_n^2(\zeta - \eta_o)}{k\ell^2} \quad , \quad (33)$$

where $\psi(x,t')$ is the particular solution of (30).

The system (30) must now be solved with $p = 0$, subject to the initial conditions (32) and (33). With $p = 0$ (30) has the solution

$$y(x,t'') = (A' + B't'')x + \sum_{n=1}^{\infty} \varphi_n(x)[A_n' \sin \frac{b_n^2 t''}{k\ell^2} + B_n' \cos \frac{b_n^2 t''}{k\ell^2}]$$

$$+ \frac{M_o\ell^2}{EI}(\frac{x^5}{5\ell^5} - \frac{12x(t'')^2}{k^2\ell^5} - \frac{x^3}{2\ell^3})$$

where $t'' \equiv t - \tau$. The constants A', B', A_n', and B_n' must be determined so as to satisfy initial conditions (32) and (33). Proceeding as before we find

$$A' = \frac{96p\ell^3}{\pi^7 EI} \left[\sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^7} (1 - \cos n^2 \eta_0) \right.$$

$$\left. + (\zeta - \eta_0) \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^5} \sin n^2 \eta_0 \right] + \frac{M_0}{EI}\left[\frac{3(\mu-8)(\zeta-\eta_0)^2}{2\pi^4} + \frac{39}{560} \right]$$

$$B' = \frac{96p\ell}{\pi^5 EIk} \left[\sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \sin n^2 \eta_0}{n^5} + \frac{\pi^3 (\mu - 8)(\zeta - \eta_0)}{32\mu} \right]$$

$$A_m' = A_m \cos \frac{b_m^2 (\zeta - \eta_0)}{\pi^2} - B_m \sin \frac{b_m^2 (\zeta - \eta_0)}{\pi^2}$$

$$B_m' = A_m \sin \frac{b_m^2 (\zeta - \eta_0)}{\pi^2} + B_m \cos \frac{b_m^2 (\zeta - \eta_0)}{\pi^2} + \frac{4p\ell^4}{EIb_m^5 (\sinh\frac{b_m}{2}+\sin\frac{b_m}{2})}$$

$$+ \frac{4p\ell^4}{EIb_m^3 (\sinh^2 \frac{b_m}{2} - \sin^2 \frac{b_m}{2})} \left[\frac{\sinh\frac{b_m}{2} - \sin\frac{b_m}{2}}{b_m^2} \right.$$

$$\left. - \frac{1}{4} \sin \frac{b_m}{2} \cosh \frac{b_m}{2} \right]. \qquad (34)$$

Now plastic deformation at the hinge will cease when the relative angular velocities of the two half beams at $x = \ell/2$ become equal, i.e., when $\left.\frac{\partial^2 y}{\partial x \partial t}\right|_{x=\ell/2} = 0$. However, after taking one x and one time derivative of the solution the eigenfunction series no longer converges. To avoid this difficulty, the slope $y_x(\ell/2,t)$ is plotted as a function of time and the maximum point

of the curve is determined by inspection.

These times are used to calculate the final mid-point deflection, which is plotted as a function of the load intensity μ for several pulse times in Fig. 10. The final central angle is plotted in the same manner in Fig. 11. The results from a plastic-rigid analysis of the beam [2] are included for comparison.

For small enough load intensity values and/or short enough pulse times, the bending moment may reach M_o after the load has been removed, or not at all. The analysis in this case is similar to that above and will not be presented here. The results are included with those in Figs. (10, 11).

III. Discussion:

The criterion developed in [1] for the validity of a plastic-rigid analysis for this problem takes the form

$$2M_o\Theta_o \gg \frac{M_o^2 \ell}{2EI} , \tag{35}$$

where Θ_o is the final central angle of the beam. Inequality (35) may also be written in the following form:

$$\frac{4\zeta^2 f(\mu)}{\pi^4} \equiv \frac{W_P}{W_E} \gg 1, \tag{36}$$

where $f(\mu)$ is the dimensionless parameter involving the final central angle shown in Fig. (11).

Figure (12) shows a plot of the differences (expressed as percent) between mid-point deflections of the present elastic-plastic solution and those given by the single-hinge and correct rigid-plastic solutions vs. W_P/W_E, for values of τ/T equal to

1/2 and 2. T is the fundamental period of vibration of the beam.

It is seen that for W_P/W_E becoming large, the single-hinge rigid-plastic solution differs very much from the correct rigid-plastic results and is presumably wrong since violations of the plasticity condition would occur at sections other than the mid-section. The approach to agreement as W_P/W_E becomes large of the present elastic-plastic solution with the single-hinge rigid-plastic solution is to be expected on the basis of the energy hypothesis for the validity of a rigid-plastic analysis: both solutions are correct (for very large W_P/W_E) for a beam strengthened so that plastic flow takes place only at the mid-section where the bending moment is equal to M_O. The agreement at large W_P/W_E values between the single-hinge rigid-plastic solution and the present elastic-plastic solution can thus be taken as an indication that the elastic-plastic results are incorrect in the range of large W_P/W_E. However, the elastic-plastic analysis gives accurate results for fairly low load intensities, where the deformation is mainly elastic and plastic deformation occurs only at a single hinge at the center of the beam.

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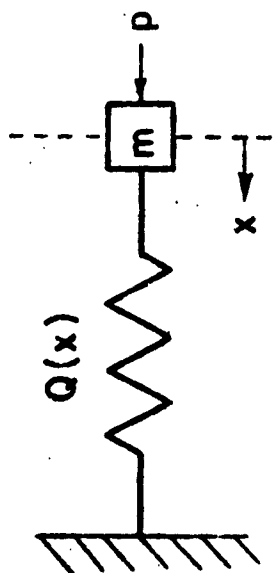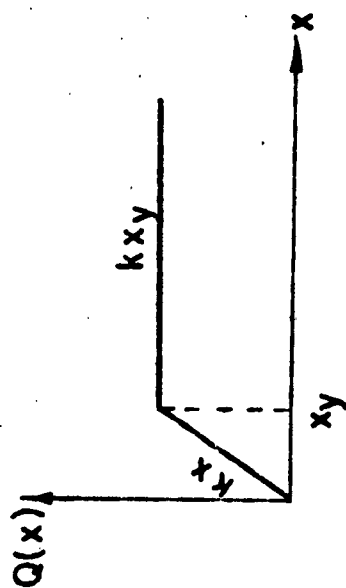
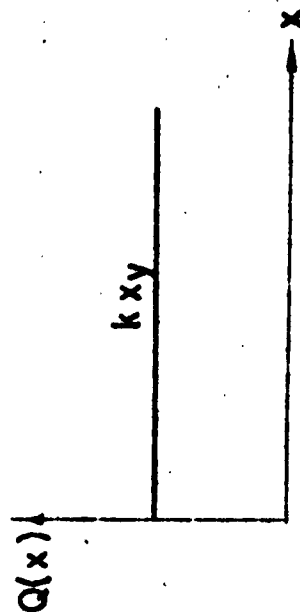Fig. 2. Spring Force Displacement
Curve: Elastic Plastic Case

Fig. 4. Spring Force Displacement
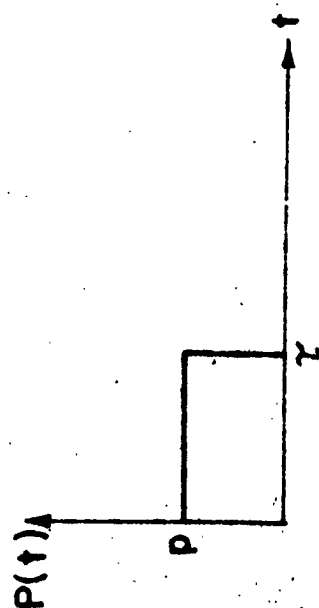Curve: Rigid Plastic Case

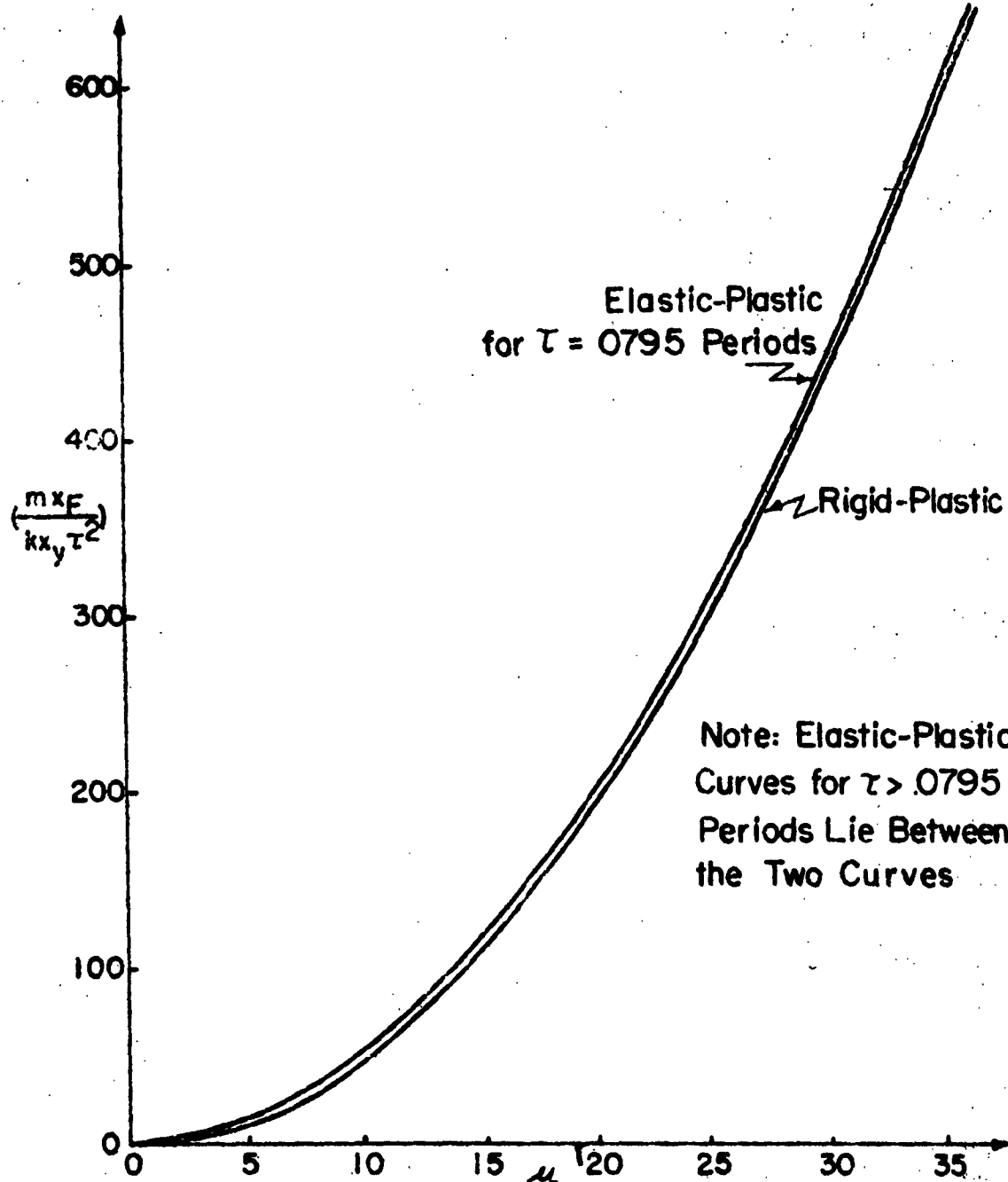Fig. 1. Mass-Spring System

Fig. 3. Load Time Curve

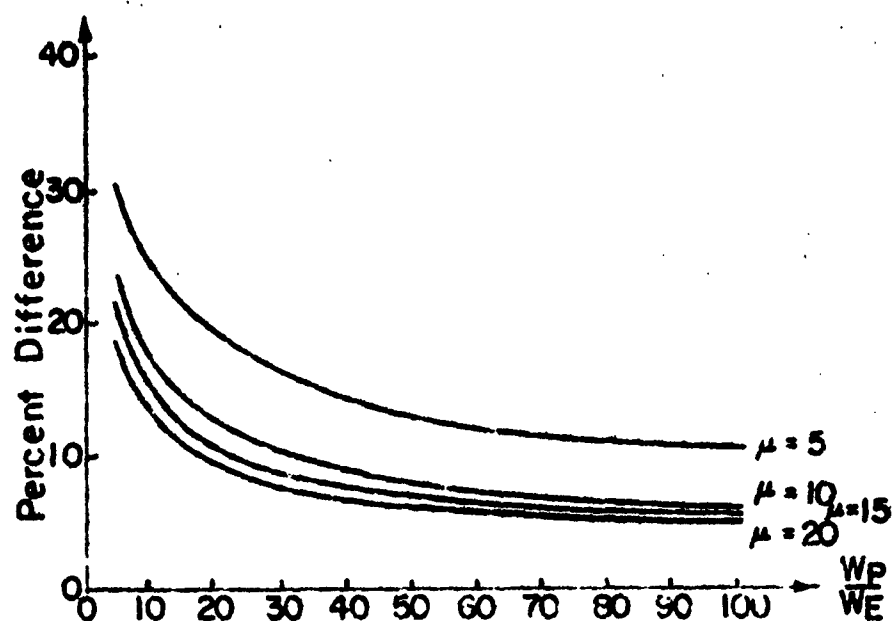Fig. 5. Final Deformation vs. Load-Mass Spring System

Fig. 6(a). Plot of % Difference ($\frac{E.P.-R.P.}{E.P.} \times 100$) vs $\frac{W_P}{W_E}$ for Various μ
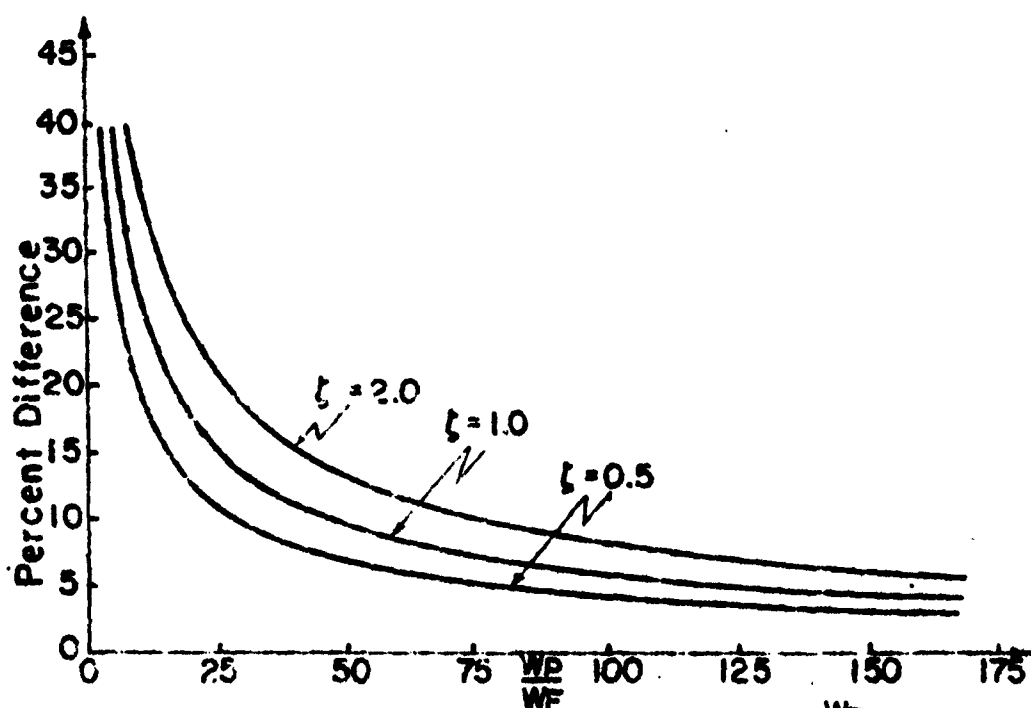

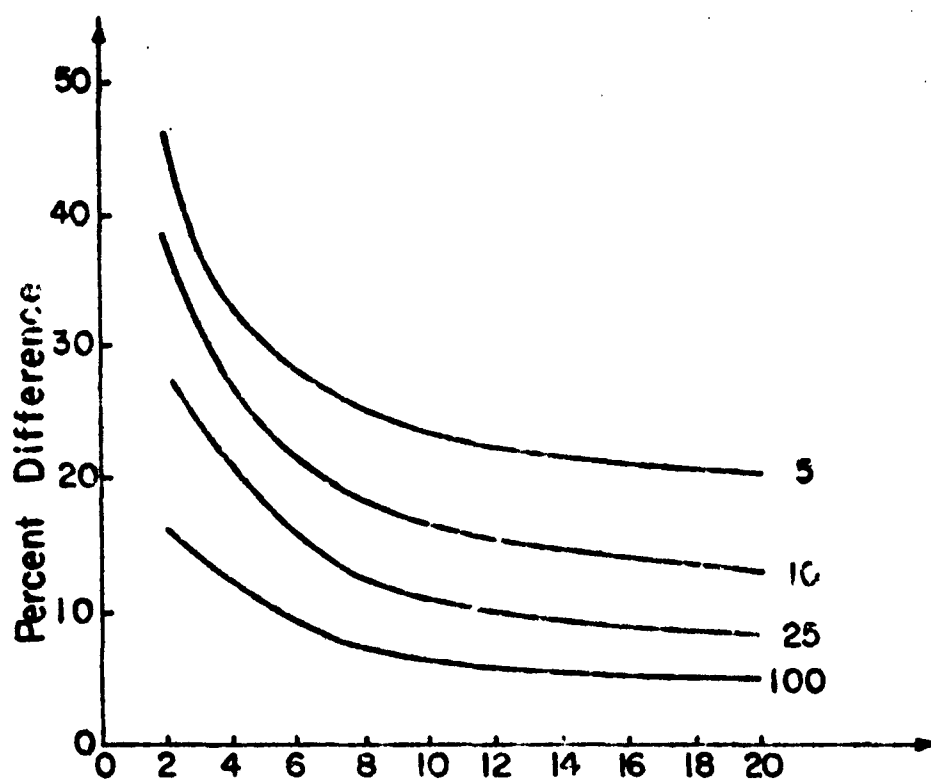Fig. 6 (b). Plot of % Difference vs. $\frac{W_P}{W_E}$ for Various ζ

Fig 6(c). % Difference $(\frac{EP\text{-}RP}{EP} \times 100)$ vs. μ
For Various $\frac{W_P}{W_E}$ values
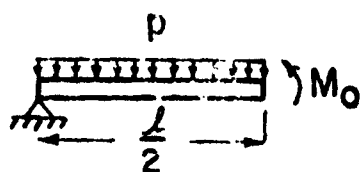
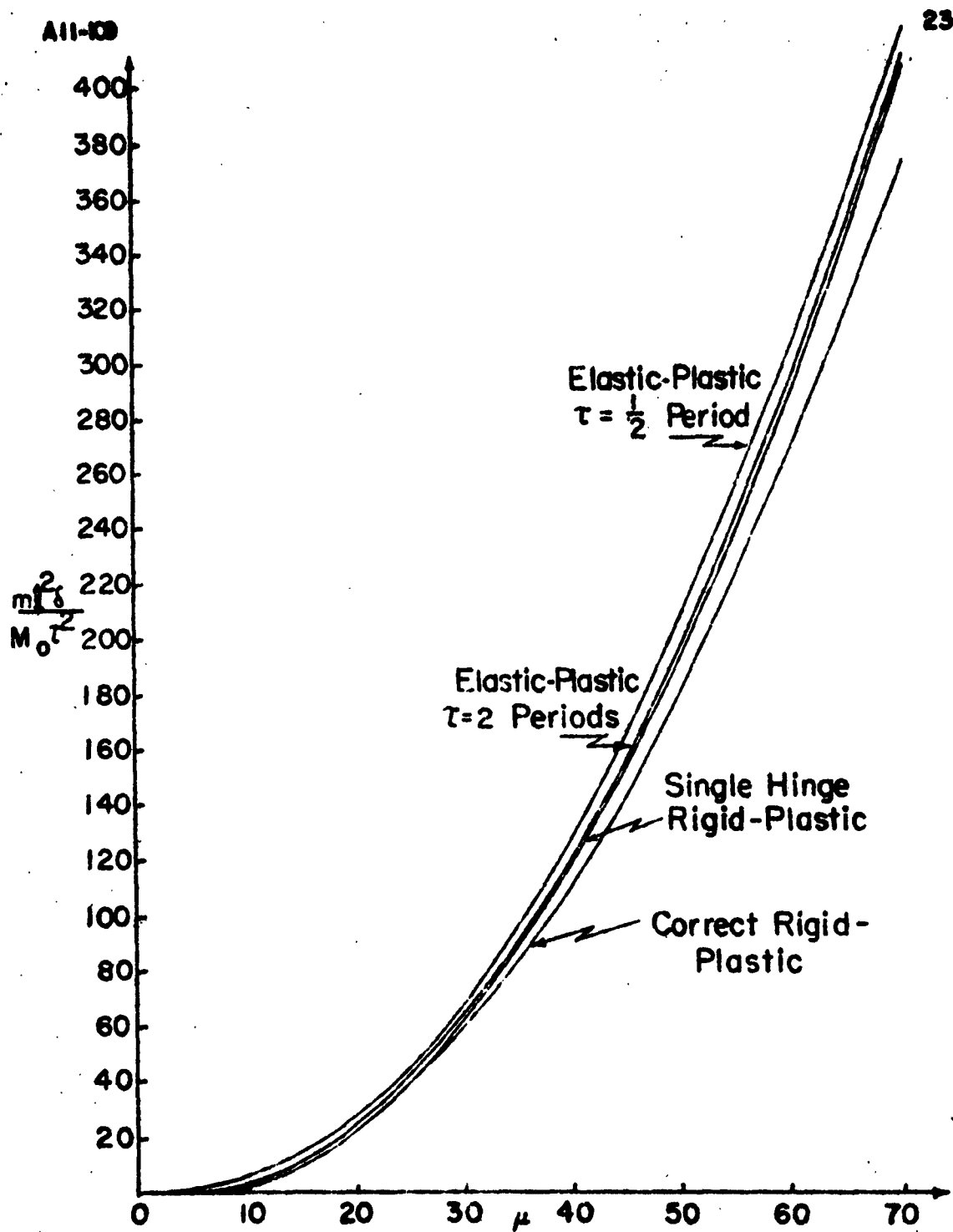

Fig.   7.

Fig. 8. Plot of η_0 vs. μ
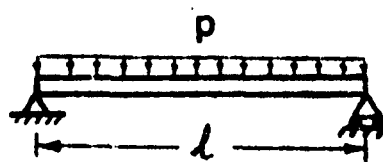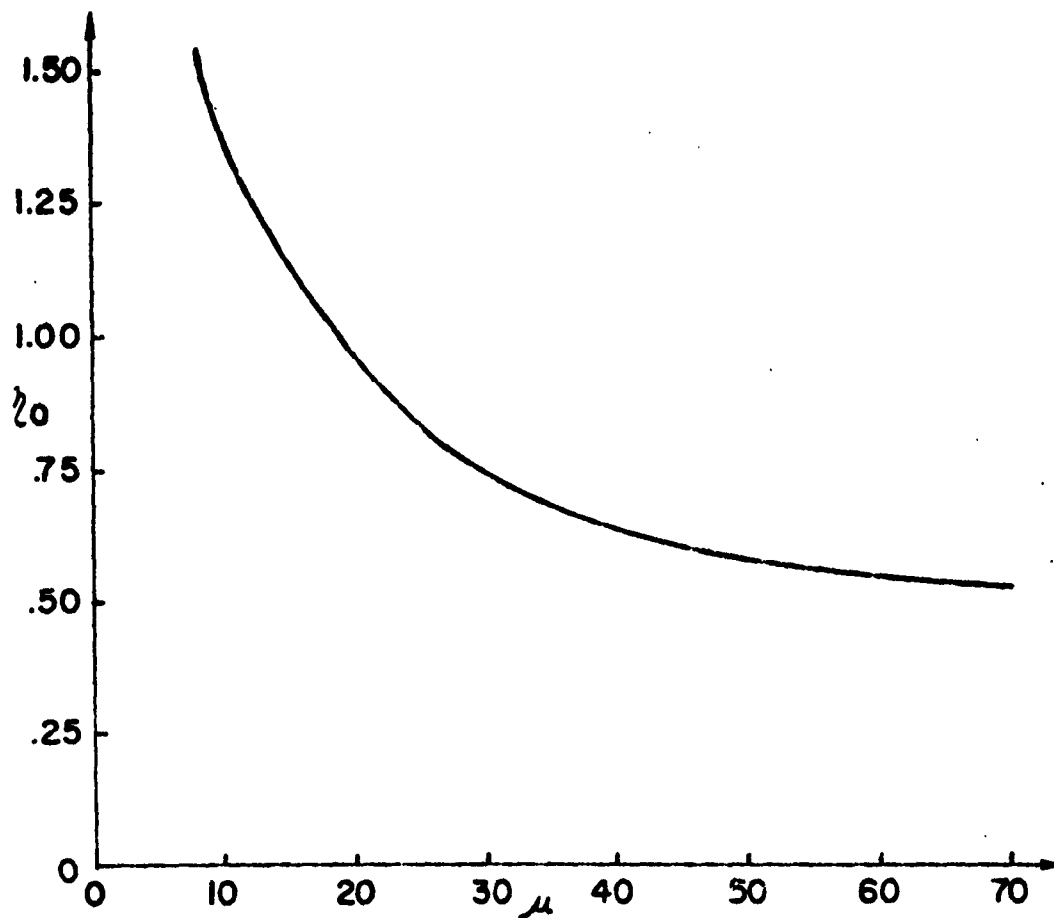
Fig. 9.

Fig. 10. Final Mid-point Deflection vs. μ

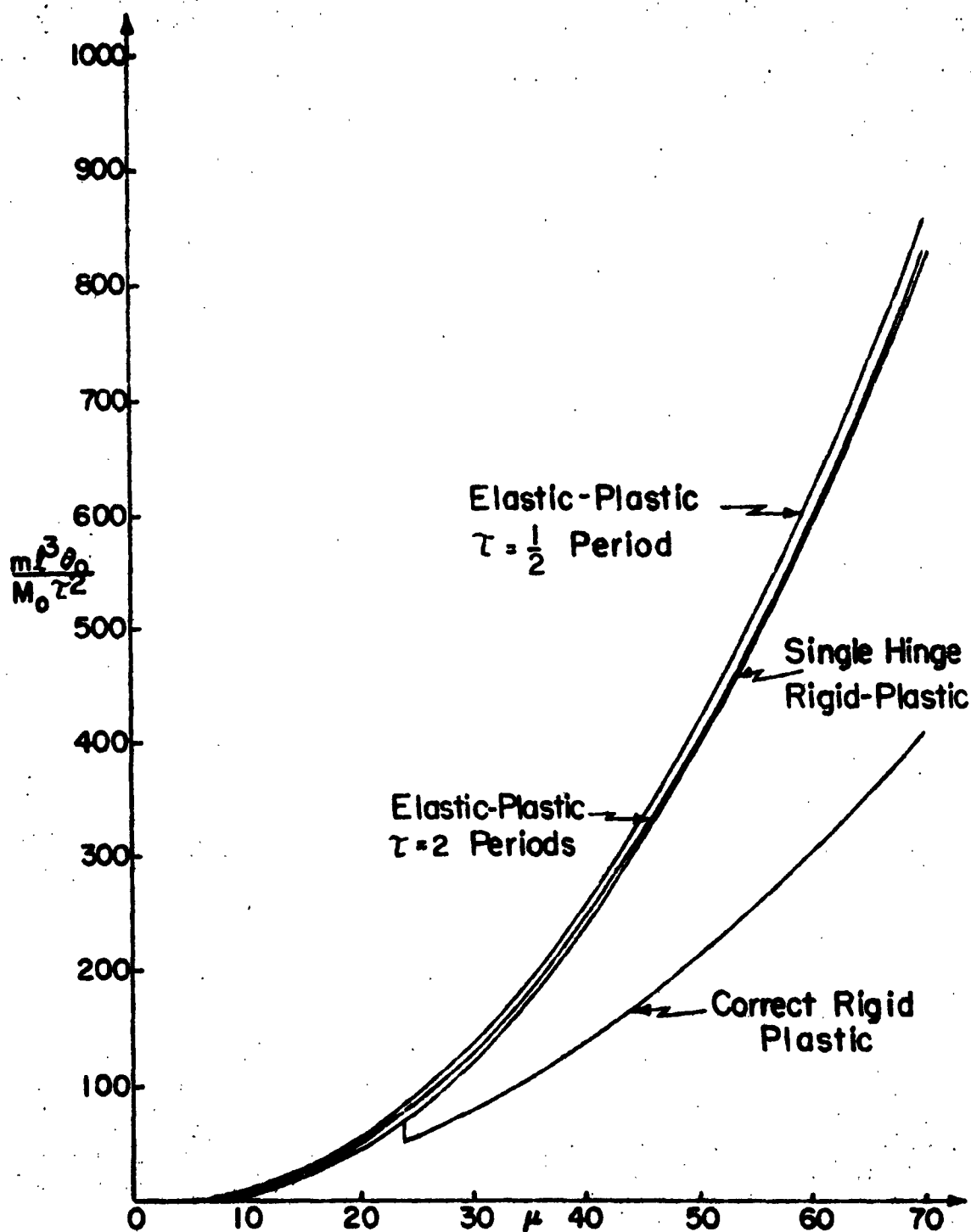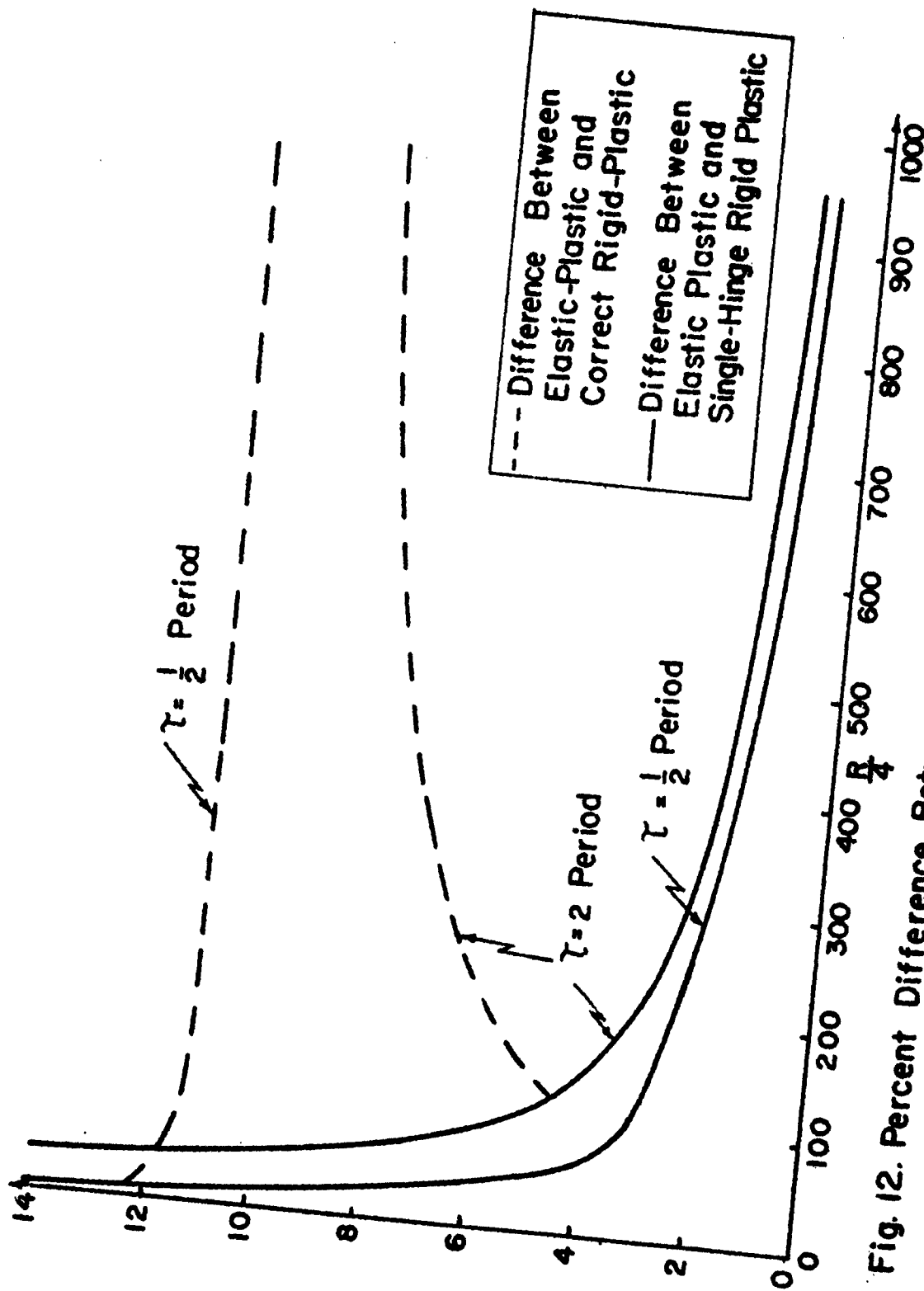Fig. 11. Final Central Angle vs. μ

Fig. 12. Percent Difference Between Elastic-Plastic and Rigid Plastic Mid-point Deflections vs. $R \equiv W_P/W_E$